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L. E. J. BROUWER AND KARL POPPER: TWO PERSPECTIVES ON MATHEMATICS

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ABSTRACT: This article provides an appraisal of Popper's criticism of L. E. J. Brouwer's intuitionist mathematics. Despite the extensive scholarship on Popper, his engagement with Brouwer's thought has largely been overlooked. Through his critical engagement with Brouwer, Popper provides an eloquent overview of some of the innovative features of his own later objectivist evolutionary epistemology. For Brouwer, the intuitional ground of mathematics completely separates mathematics from mathematical language. Intuitionistic mathematics is an essentially languageless activity of the mind. The mathematician is prioritised as the ultimate source of authority over the formalised representation of mathematics. Popper, while appreciating the important role of intuitions in identifying problems and deriving solutions was highly critical of Brouwer's subjectivist orientation. Popper reconstructed the problem of intuition in terms of evolutionary cognition as the problem of "unconscious expectations" or "background knowledge". This background knowledge does not derive from some pristine source of truth in the subject, but is the result of previous problem solving attempts, which become built into our cognitive apparatus and unconsciously inform our actions in the form of *conjectures*. What is crucial for Popper, is the way we externalise our knowledge in the form of conjectures, which both enables it to be criticised, as well as potentially lay bare hitherto unseen implications. For Popper, this was crucial to the way knowledge grows, and is necessary for the development of the self, which is dependent upon linguistic communication.

KEYWORDS: Mathematics; Popper; Brouwer; Evolutionary Epistemology

INTRODUCTION

This article provides an appraisal of Popper's criticism of L. E. J. Brouwer's intuitionist mathematics.[1] Despite the extensive scholarship on Popper, his engagement with Brouwer's thought has largely been overlooked. Through his critical

[1] I would like to thank Professor Wayne Hudson for reading drafts of this paper and for providing invaluable criticisms and suggestions.

engagement with Brouwer, Popper provides an eloquent overview of some of the innovative features of his own later objectivist evolutionary epistemology. A direct critique of Brouwer's thought presented in Popper's lecture "Epistemology Without A Knowing Subject" (1967) published in *Objective Knowledge: An Evolutionary Approach* (1972), however, influence of his engagement with Brouwer on the problem of intuitionism is recurrent in his latter writings. Both Popper and Brouwer agreed that the problem of sequential or rational reasoning and its relation to our immediate perceptions is a crucial problem for human knowledge. For Popper, language or sequential rationality distorts our unconscious immediate perceptions, which comprise the vast bulk of our knowledge at any given time. Brouwer on the other hand, viewed a greater private accessibility of such intuitions or immediate perceptions. For Brouwer, the intuitional ground of mathematics completely separates mathematics from mathematical language. Intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. The mathematician is prioritised as the ultimate source of authority over the formalised representation of mathematics. In this context, language, including mathematical language is incapable of providing the means of communicating them to others. Popper, while appreciating the important role of intuitions in identifying problems and deriving solutions was highly critical of Brouwer's subjectivist orientation.

For Popper, the problem of intuition was re-construed in terms of evolutionary cognition as the problem of "unconscious expectations" or "background knowledge". The way this relates to our discursive knowledge is not restricted to mathematics or the hard sciences but is crucial in understanding all instances of human problem solving. This background knowledge does not derive from some pristine source of truth in the subject, but is the result of previous problem solving attempts, which become built into our cognitive apparatus and unconsciously inform our actions in the form of *conjectures*. What is crucial for Popper, is the way we externalise our subjective knowledge in the form of conjectures, which both enables such knowledge to be criticised, as well as potentially lay bare hitherto unseen implications. For Popper, this was crucial to the way knowledge grows, and is necessary for the development of the self, which is dependent upon linguistic communication. Section 2 provides a brief overview of Brower's intuitionist mathematics relevant to the current discussion. Section 3 then provides a close reading of Popper's analysis and response to Brouwer.

BACKGROUND

Brouwer's intuitionism is related to conceptualism, which holds that abstract entities exist only insofar as they are constructed by the human mind. Thus, for the intuitionist

mathematician the abstract entities, which occur in mathematics, such as sequences or order-relations, are all mental constructions. As a result, the amount of abstract entities for an intuitionist mathematician is greatly restricted in comparison to classical mathematics as well as in logicism. Popper did not agree with mathematical logicism, rather he agreed with the "realist" or Platonist doctrine that abstract entities have an existence independent of the human mind. This view is justified on the basis that mathematics is full of abstract entities such as numbers, functions and sets to name a few, which according to Plato exist outside the mind. From this we get the medieval philosophical doctrine of realism held that the mind discovers such entities but does not create them. Realism allows us to accept many more abstract entities in mathematics than a philosophy that had limited us to accepting on those entities the human mind can construct. Russell was a realist and accepted the abstract entities that occur in classical mathematics without questioning whether our own minds can construct them. This is the fundamental difference between logicism and intuitionism, since in intuitionism abstract entities are admitted only if they are man made. Brouwer himself described the intuitional ground of mathematics in the following words:

Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognising that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics. [2]

Brouwer in effect founded the mathematical philosophy of intuitionism as a challenge to the then-prevailing formalism of David Hilbert and his collaborators including Paul Bernays, Wilhelm Ackermann and John von Neumann. As a variety of constructive mathematics, intuitionism is essentially a philosophy of the foundations of mathematics. The position of intuitionism has traditionally maintain that the foundations of mathematics lie in the individual mathematician's intuition, the result is that mathematics into an intrinsically subjective activity. Intuitionist mathematics holds that mathematics should be defined as a mental activity and not a set of theorems. It is an activity by which we carry out one after another in a series of sequences those mental constructions inductively. Popper was emphatic that Brower's intuitionism raised some crucial issues for mathematics.

[2] Brouwer, "Historical background, Principles and Methods of Intuitionism", p. 141.

Brouwer's understanding of intuition for the foundations of mathematics is more modest in what it claims to accomplish than the term intuitionism suggests. Brouwer's project was a continuation of a common paradigm of a philosophical conception of mathematical intuition derived from Kant and his argument that mathematical intuition concerns space and time as forms of our sensibility.[3] Brouwer took issue with certain rules of classical logic, particularly the law of the excluded middle. He also warned of the tendency to think of mathematics simply in terms of mechanical deductions as this excluded the important creative role of imagination which he viewed as a defining characteristic of a good mathematician. For Brouwer, the mental life of the mathematician lay at the heart of mathematics, and from an instrumentalist perspective, formal notation and the mechanical application of rules are "merely" imperfect ways of communicating the true heart of mathematics. According to Placek, Brouwer:

> …runs against the substantial part of the mathematical tradition with his sharp separation between, on the one hand, intuitive mathematics and its creator as well as the ultimate authority on what holds or is certain in intuitive mathematics, and on the other hand, the linguistic representation of mathematics with laws of logic understood as idealized regularities obtaining between reports on mathematical activities of the subject.[4]

In this respect Brouwer's view of mathematics provides a radical alternative to that of Hilbert and his fellow formalists who emphasised the proper use of symbols. The formalists argued that the systematic, rule-governed use of symbols is central to the meaning of mathematical practice. What I contend, is that Popper appeared to be mediating a kind of position in between these two views. He appeared to do this by acknowledging the importance of intuition or what he termed "background" or dispositional knowledge in informing our conjectures. He also agreed with the formalists on the issue of the importance of mathematics systemised symbolic rendering as only by such objective denotation can our knowledge be communicated, criticised and grow; in this way mathematical problems take on a life of their own. This second point was crucial to Popper's theory of objective knowledge. Popper had a problem with this and saw Brouwer's intuitionism as proposing an interesting solution to this problem by providing an account of the way we create such abstract entities. Contrary to what Brouwer believed, Popper argued that we can indeed just pick out number 3 out of the sky without going through the mental steps of constructing 1 and 2 after initially constructing (inventing) natural numbers. A great mediator, Popper

[3] Parsons, Charles. "Platonism and Mathematical Intuition in Kurt Gödel's thought", p. 44.
[4] Ibid., p. 101.

sought an approach to mathematics that navigated between logicism (formalism) and intuitionism in a way that both avoided excessive Platonism of the logicist and the "subjective" constructive foundationalism of intuitionism.

3. POPPER'S RESPONSE TO BROUWER'S INTUITIONIST MATHEMATICS

I will now provide an overview of Popper's objectivism in the form of his World 3 thesis which will help to frame where Popper was saw his 'objectivist epistemology' as a point of difference, indeed alternative counter position to Brouwer's subjectivism. Popper's contribution in the co-authored *The Self and Its Brain: An Argument for Interactionism* (1977) with John Eccles was such an attempt to show the cognitive conditions for objective knowledge by relating his metaphysical "World 3 pluralism" to the latest developments in cognitive and neurological science. The theory of the three worlds or realms was essentially an expansion upon his earliest work in developmental cognitive psychology in which he integrated a Selzian active problem-solving mind with Karl Bühler's evolutionary theory of language function. Popper's 'World 3 Thesis' is a "metaphysical research program" which delineates three realms of information transference that is a crucial feature of his "evolutionary epistemology". This explanatory model differentiates between the physical world (World 1), our individual subjective cognition (World 2) and the moderate or evolutionary Platonism of the "autonomous" exosomatically evolved realm of ideas, theories, arguments and thought products (World 3). This metaphysical system functions as a heuristic for explaining the semi-autonomous nature of the products of our cognitive activities and how these products 'feed-back' into Worlds 2 and 1.[5]

It is a mistake to argue that Popper did not recognise the crucial role that "intuition" or immediate perceptive awareness plays in our cognitive functioning. Popper agrees in *The Open Society* with the thesis he associates with Aristotle and Plato that "we possess a faculty, intellectual intuition, by which we can visualize essences and find out which definition is the correct one". However, following Kant, Popper modifies this with a scepticism of this kind of knowledge. The result is that Popper admits that we possess something, which may be described as "intellectual intuition". Popper's use of intuition here is in keeping with the German and Kantian term

[5] For a detailed discussion on Popper's World 3 thesis see: Naraniecki. *Returning to Karl Popper: a reassessment of his politics and philosophy*, Chapter 6. Donald Gillies has outlined in a lecture that Popper "seems to have developed his theory of world 3 in the second half of the 1960s. What became the standard account of this theory appeared as chapters 3 and 4 of Popper's book *Objective Knowledge*, which was published in 1972. Chapter 3 *Epistemology Without a Knowing Subject* had already been published in 1968, while Chapter 4 *On the Theory of the Objective Mind* was based on articles, which had appeared in 1968 and 1970. In his 1972 exposition, Popper uses the term 'third world', but later, following a suggestion by his friend Sir John Eccles, he preferred to use the term 'world 3'." See: Gillies, "World 3 and the Internet".

Anschauung in the sense of that knowledge which is immediately given in experience yet are not ideas, thoughts, abstractions or generalisations. This is also where Brouwer's intuitionism derives from, as intuition is understood in the Kantian sense of "immediate awareness" of time from which mathematical concepts can be securely constructed upon. The mental construct of one natural number after another would not have been possible did we not possess an awareness of time within us in a way that we might now refer to as "in-built" knowledge. This differs from the English term intuition which has the same meaning as the German word *Intuition*, which means an inexplicable kind of direct information from some supernatural source associated with mystical experiences at the moment when though, as Goethe's Mephistopheles describes "All Heaven and Earth in rapture penetrating".

Popper summarises his view concerning Brouwer's intuitionist mathematics in the paper "Epistemology Without a Knowing Subject" (1967), published in *Objective Knowledge* (1972).[6] Here, Popper begins his engagement with Brouwer's mathematics by locating the problem in the work of Kant that "*intuition is a source of knowledge*; and 'pure' intuition ('the pure intuition of space and time') is an unfailing source of knowledge: from it springs *absolute certainty*".[7] Popper rightly notes the importance of Brouwer's adoption of this epistemological doctrine from Kant. For Popper it was Kant who upheld the doctrine (against Descartes) that we do not possess a faculty of intellectual intuition, therefore reason, that is, our intellect, which includes our concepts, remain empty or analytic, "unless indeed they are applied to material which is either given to us by our senses (sense intuition), or unless they are '*concepts constructed in our pure intuition of space and time*'". Only in this way for Kant can we obtain synthetic knowledge *a priori* as (importantly for Popper) *our intellect is essentially discursive* as it is bound to proceed by logic, which is empty – 'analytic'.[8]

What Kant had ingeniously identified here was an objective order of thought. We have a faculty of pure intuition which comes first, however this is strictly limited, and distinct from our self-conscious way of thinking which is discursive. Popper states that for Kant:

> …sense intuition presupposes pure intuition: our sense cannot do their work without ordering their perceptions into the framework of space and time. Thus space and time are prior to all sense-intuition; and the theories of space and time –geometry and arithmetic – are *a priori* valid. The source of their *a priori* validity is

[6] Brouwer when asked whom of his friends he would like to be invited for the traditional tea party of the Provost included Popper in the list. See: van Dalen. 2005. *Mystic, Geometer, and Intuitionist: The Live of L. E. J. Brouwer*, p. 864.
[7] Popper, *Objective Knowledge*, p. 130.
[8] Ibid., p. 130.

> the human faculty of *pure intuition*, which is strictly limited to this field, and which is strictly distinct from the intellectual or discursive way of thinking.[9]

However there is a problem with Kant's assertion that geometry is a priori valid (a pure intuition) rather than a product of rational argumentation as "Euclid's geometry, whether or not it uses pure intuition, certainly makes use of intellectual argument, of logical deduction. *It is impossible to deny that mathematics uses discursive thought.*"[10] This is where Brouwer comes into the discussion via his theory of *the relation between mathematics on the one hand and language and logic on the other*. For Popper, Brouwer:

> …solved the problem by making a sharp distinction between *mathematics as such* and *its linguistic expression and communication*. Mathematics itself he saw as an extra-linguistic activity, essentially an activity of mental construction on the basis of our pure intuition of time. By way of this construction we create in our intuition, in our mind, the objects of mathematics which afterwards – after their creation – we can try to describe, and to convey to others. Thus the linguistic description, and the discursive argument with its logic, come after essentially mathematical activity: they always come after an object of mathematics – such as a proof – has been constructed.[11]

For Brouwer, according to Dirk van Dalen, as long as mathematics is considered as the science of space and time, it belongs to the field of activity of logical reasoning. This was the case not only in the days when space and time were believed to exist independently of human experience, but also when as a result of Kant they had been taken for innate forms of conscious exterior human experience. Popper, like Kant, also saw the intellect as being inherently structured, however this structure was of a hidden logical and evolved dispositional kind, which underpins our discursive operations. Traditionally, as part of the field of logical reasoning mathematics continued to be associated with the conviction that its assertions are either false or true, whether we know it or not and that even with the extinction of humanity, mathematical truths, just as laws of nature will survive. This approach to mathematics for Brouwer was not in keeping with Kant's view of time as an innate form of consciousness. According to can Dalen:

> Only after intuitionism had recognized mathematics as an autonomic interior constructional mental activity, which although it has found extremely useful linguistic expression and can be applied to an exterior world, nevertheless neither in its origin nor in the essence of its method has anything to do with language or an exterior world, on the one hand axioms became illusory, on the other hand

[9] Ibid., pp. 130-131.
[10] Ibid., p. 131.
[11] Ibid., p. 132.

> the criterion of truth or falsehood of a mathematical assertion was confined to mathematical activity itself, without appeal either to logic or to a hypothetical omniscient being.[12]

Popper recognized Brouwer's move here in solving the problem which in Kant's *Critique* at first sight appears to be a contradiction by sharply distinguish between two levels, "one intuitive and mental and essential for mathematical thought, the other discursive and linguistic and essential for communication only."[13] Popper also recognizes the importance of Boruwer's idea that mathematical thought objects arise in human mind or what Popper called the second world (or World 2). Popper states that the objectivity of mathematical objects for Brouwer, that is, their character as objects, and possibility of their existence "rested entirely in the possibility of repeating their construction at will." According to Popper, "for the intuitionalist, mathematical objects existed in the human mind; while for the formalist, they existed 'on paper'". So mathematical objects are different from language, they arise intuitionally in the mind (World 2) then we try to grasp them via discursive reasoning.

Crucially however Popper opposes Brouwer by adopting a moderate Platonism, which gives mathematical objects "autonomous" status in World 3, and thus their "criticisablity" in discursive argumentation. This is where Popper diverges from Brouwer's "subjectivism" as Brouwer's "mentalism" for Popper leads to his rejection of 'Platonism', by which Popper understand the doctrine that mathematical objects have what he called an 'autonomous' mode of existence in which it can be said that "they may exist without having been constructed by us, and thus without having been proven to exist."[14] Popper, however following Kant understood that the counterfactual idea, in this case the "autonomous mode of existence" gives us a metaphysics that can provide theories that better order empirical reality. It is as a result of this critical purchase of his moderate Platonism that he maintained his ontological World 3 Thesis even after strong criticism that everything in it can be attributed to mental (World 2) states.[15]

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[12] van Dalen. 2005. *Mystic, Geometer, and Intuitionist: The Live of L. E. J. Brouwer*, p. 864.
[13] Popper, *Objective Knowledge*, p. 132.
[14] Ibid., p. 134.
[15] For O'Hear's criticism of Popper here see: O'Hear, *The Arguments of the Philosophers: Karl Popper*. On Page 183 he states: "My criticism of Popper will be primarily to show that everything he wants to say about unintended and unforeseen consequences and the challenge of World 3 can be explained in World 2 terms, and that the idea of theories completely autonomous of human behaviour is misleading." However, I argue that World 3 is necessary in order to develop theories concerning the actual way discoveries are made, problem solving occurs and creative is possible. Without World 3, if we rely on a more observationally "realistic" theory, different theories and explanations would have to be found, ones that perhaps would not provide as satisfactory explanations or theories of these processes. The counter-

Popper argued that Kant's sharp distinction between intuition and discursive thought cannot be upheld. 'Intuition', for Popper, whatever it may be, is largely the product of our cultural development, and of our efforts in discursive thinking. Popper viewed Kant's idea of one standard type of pure intuition shared by us all can no longer be accepted. As we have trained ourselves in discursive thought, our intuitive grasp becomes utterly different from what it was before.[16] Thus by discussing our intuitions we alter them, they become something else, as there is a meta-level transformation. This argument raises interesting implications for political philosophies of authenticity, which are based upon the immediate accessibility of our intuitions about essential cultures and practices and seek to build an identity politics upon this.

According to Popper "this [intellectual intuition] can never serve to establish the truth of any idea or theory, however strongly somebody may feel intuitively that it must be true, or that it is 'self-evident'."[17] The reason Popper gives for this becomes apparent when we look at his response to Kant's thesis on pure intuition. Central to overcoming Kant's transcendental idealism was the need to overcome his intuitionist geometry with its understanding that we have an *a priori* pure intuition of space and time. The way Popper overcame this must be seen within their proper evolutionary context. In *Epistemology Without a Knowing Subject* a lecture given in 1967 Popper identifies the problem as originating with Plotinus's distinction between *intuition*, which is God's way of knowing everything in a flash, timelessly and *discursive* thinking, by which humans argue step by step.[18] However, he then criticizes this position that he also associates with Kant's subjectivism. 'Intuition' (*Anschauung*), is largely the product of our cultural development, and of our efforts in discursive thinking." The idea of one type of pure intuition shared by all can no longer be accepted: "For after having trained ourselves in discursive thought, our intuitive grasp becomes utterly different from what it was before".[19] However, for Popper intellectual intuition as inbuilt disposition or tendencies are a product of our "background" knowledge of which we are only ever consciously aware of a miniscule fraction at any point in time. It is this background knowledge that plays a governing role in our decision to conjecture a hypothesis of which we only later after the decision is made, hazard a rational justification for our choice. Discursive rationality comes in at the point of critical

intuitive and non-visualisable metaphysics of Popper paradoxically provide give rise to better theories for empirically describing actual processes of creativity and discovery than the more empirically grounded theories of mental activity.

[16] Popper, *Objective Knowledge*, p. 135.

[17] Popper, *The Open Society and Its Enemies*, vol. 2, pp. 15-16.

[18] Popper, *Objective Knowledge: An Evolutionary Approach*, p. 131.

[19] Ibid., p. 135.

reflection of our conjectures, that is, at the attempt of trying to (erroneously) justify them or alternatively, ideally attempt to refute them.

Understanding what Popper meant by rationality is crucial to understanding the concessions and restrictions he was willing to place on intuitionism. This is most clearly spelled out in the collection of lectures *Knowledge and the Mind-Body Problem: in Defence of Interaction* (1994):

> …I mean by 'rationality' simply a critical attitude towards problems – the readiness to learn from our mistakes, and the attitude of consciously searching for mistakes and for our prejudices. Thus, I mean by 'rationality' the attitude of conscious, critical, error elimination…Yet our knowledge consists of a huge amount of dispositions, expectations, and theories of which only a small number can be placed consciously before us at a particular moment of time…at any moment there will be only one theory before us that is selected for critics…But a huge amount of knowledge, knowledge on all kinds of levels of importance, is used, mostly unconsciously, in the investigation of any theory. I have called this knowledge 'background knowledge'.[20]

Popper's view was that discursive thought (understood as a sequence of linguistic arguments) "has the strongest influence upon our awareness of time and upon our awareness of the development of sequential order". Thus Kant's great "illumination", the appeal to two structurally different types of representation that the mind uses; the pictorial-like component of representation or intuition, and the sentence-like component of representation or concepts are no longer distinguished as separate mental faculties. For Popper, Kant's idea of the pure intuition of time is naturalized, as our awareness of sequential time is no longer seen as a pure intuition but a product of language and discursive reasoning which gave rise to the notion of sequencing, numbers and mathematics which inturn supplied the basis for the erroneous view that sequential time was a pure intuition.[21] As a result of our cultural and discursive lens, what we may experience as pristine intuitions are laden with discursive sedimentation. Rather than seeing both aspects of representation (the conceptual and the intuitional) sided by side as co-operating faculties of representation, Popper as a result of his early work in cognitive psychology, provide an alternative view on how concepts and intuitions fit together in the mind's activity. Popper's argument for the hypothetical-deductive character of our rational knowledge posits a kind of feed-back loop between

[20] Popper, *Knowledge and the Mind-Body Problem: in Defence of Interaction* (1994), p. 134.

[21] It is not that we do not have an interior experience of time as Popper argued that all live has an awareness of its own built-in or internal clock. Rather it is the kind of experience of time as a division into sequential standardised units that he was arguing against. See: Popper, *Knowledge and the Mind-Body Problem: In Defence of Interaction*, p. 130.

hypothetical pre-linguistic intuitions such as our built-in embodied or acquired and dispositional knowledge and our *a posterori deductive* discursive rationalisations, associated with our attempts to articulate them with the latter often clouding and colouring the former in ways we are often not conscious of. For Popper:

> …though originally constructed by us—the third world originates as our product—the thought contents carry with them their own unintended consequences. The series of natural numbers which we construct creates prime numbers—which we *discover*—the most fertile citizens of the third world—and these in turn create problems of which we never dreamt. This is how mathematical discovery becomes possible. Moreover, the most fertile citizens of the third world—are *problems*, and new kinds of *critical arguments*. Thus a new kind of mathematical existence emerges: the intuition which makes us see problems, and which makes us understand problems prior to solving them.[22]

Language and discursive thought interact with more immediate intuitive constructions, thereby destroying the "ideal of absolute evidential certainty which intuitive construction was supposed to realize". The evolution of language, particularly discursive communication involving sequences of arguments, led to the emergence of mathematics which manipulates our intuitions, particularly our intuition of time. Our intuitions are influenced by the "fertile citizens" of this Platonic realm of mathematics in a way that indicates for Popper that, "*The autonomy of the third world is undeniable.*"[23] Thus, as human culture via language and our efforts at discursive thinking develops, so does the kind of intellectual intuition we are capable of having evolves with our engagement with the autonomous objects of the third world.

In support of this Popper uses an example from Heyting's "grades of interest". According to the scale of grades $2 + 2 = 4$ is the highest grade of evidence whereas $1002 + 2 = 1004$ belongs to a lower grade. This can be shown not by actual counting, but by *reasoning* if natural number n is constructed, then we can effect the construction expressed by $(n+2) + 2 = n + 4$. For Popper, this analysis highlights the "unavoidable interplay between intuitive construction and linguistic formulation which necessarily involves us in discursive – and therefore logical – reasoning". It follows that we really cannot have immediate intuitive constructions for mathematics that are separate from reasoning.

Popper went even further in his criticism of Brouwer by opposing the initial construction of *twoity* from an intuition of sequential time. As the idea of a 'sequence' for Brouwer according to Popper is based upon the intuition of time, and a

[22] Popper, *Objective Knowledge*, p. 138.
[23] Ibid., p. 135.

construction based upon this intuition, signs and symbols were needed to carry out this construction. Popper, reverses this order by arguing that discursive though, that is, sequences of linguistic arguments has the strongest influence upon our awareness of time and upon the development of our intuition of sequential order. For Popper, this did not clash with Brouwer's constructivism but it did clash with his subjectivism as the "objects of mathematics can now become citizens of an objective third world: though originally constructed by us – the third world originates as our product – thought contents carry with them their own unintended consequences."[24] For Popper, although initially constructed by us, the products of mathematics are owing to reasoning.[25] Popper's position is a synthesis of the realism of logicism with constructivism, evidence in his particular moderate Platonism of World 3. We create or construct World 3 object such as natural numbers, which gives rise to the possibility of discovering things within them that no individual has invented, such as prime numbers.[26] Popper argued that geometry, physics or indeed any with all the arts or sciences has to be started by us but that once started it produces its own problems: "…we find a problem in the same way in which we find a mountain or a river, meaning by this that somehow we find something that is *there*".[27] Centrally for Popper:

> …the objects of mathematics can now become citizens of an objective third world: though originally constructed by us – the third world originates as our product – the thought contents carry with them their own unintended consequences. The series of natural numbers which we construct [W2] creates prime numbers-which we *discover*-[W3] and these in turn create problems of which we never dreamt.[28]

This belief in the discursive (theory centred) character of human reason would become one of Popper's fundamental beliefs and is the basis for his philosophical concern for the particular needs of rational argumentation in relation to the complexity of intellectual and social problems that we face. Later, in *Epistemology Without a Knowing Subject* a lecture given in 1967 and published as a chapter in *Objective*

[24] Ibid., p. 138.

[25] In contemporary physics this problem is most readily seen in relation to quantum field theory there the term "field" is so radically different from the classical conception that even theoretical physics admit they can barely visualise it. The symbolism does not give an intuitive vision of what is actually there. For a discussion of some of the philosophical problems associated with quantum field theory see: Kuhlmann, *The Ultimate Constituents of the material world: in search of an ontology for fundamental physics*.

[26] Popper, *Objective Knowledge*, p. 137. For Popper, Brouwer unlike the classical logicists realized that there was plenty wrong with classical mathematics and reacted to the idea that mathematics was nothing but a formal language game and that there were no extra-linguistic mathematical objects.

[27] Popper, *Knowledge and The Mind-Body Problem: In Defence of Interaction*, p. 36.

[28] Ibid., p. 138.

Knowledge, Popper stated that, "after having trained ourselves in discursive thought, our intuitive grasp becomes utterly different from what it was before".[29]

To understand this better it is necessary to appreciate Popper's "theory of full consciousness and of the ego or the self" which he outlined in *Knowledge and The Mind-Body Problem*:

> Full consciousness is anchored in world 3 – that is, it is closely linked with the world of human language and of theories. It consists mainly of though processes. But there are no thought processes without contents, and thought contents belong to world 3.

The self, or the ego, is impossible without intuitive understanding of certain world 3 theories and, indeed, without intuitively taking these theories for granted. The theories in question are theories about space and time, about physical bodies in general, about people and their bodies, about our own particular bodies as extending in space and time, and about certain regularities of being awake and being asleep. Or to put it in another way, the self, or the ego, is the result of achieving a view of ourselves from outside, and thus of placing ourselves into an objective structure. Such a view is possible only with the help of a descriptive language.[30]

Here we can see in the first two theses the way full consciousness is dependent upon the self's engagement with the mind independent objects of World 3 and that such theories become built-into the human organism and take the form of unconscious embodied knowledge. For Popper, developing our linguistic, communicative and discursive capabilities was crucial to forming a "self" which was a kind of teleological goal of humans and crucial for both individual as well as social flourishing. Harking back to his early work with Karl Bühler, Popper argued in *Unended Quest* that we have a disposition to interpret signals. Further, he states that there are inbuilt dispositions of great variety and complexity which he states cooperate in this field such as the disposition to love, sympathise, emulate movements, to control and correct the emulated movements, the disposition to use them and to communicate with their help, to react to language, receive commands, requests, admonitions, warnings and to

[29] Ibid., p. 36. Also see: Popper, *Objective Knowledge*, p. 135.

[30] Popper, *Knowledge and The Mind-Body Problem: In Defence of Interaction*, pp. 114-115. It is interesting here that while Popper sees the *self* as the product of viewing ourselves from *outside* by "plugging into" the discursive rationality of world 3, Brouwer was concerned with the *soul* and its return to its deepest home by avoiding sequential reasoning in favour of a "free unfolding" of "introspective harmonies". For Brouwer's mysticism see: van Dalen, Dirk. 2005. *Mystic, Geometer, and Intuitionist: The Live of L. E. J. Brouwer*. Clarendon Press: Oxford. Vol. II, p. 864. Also see: van Atten, Mark and Robert Tragesser. 2003. "Mysticism and mathematics: Brouwer, Gödel, and the common core thesis", in *Klarheit in Religionsdingen*, Leipziger Universit**a**tsverlag Leipzig, 145-160.

produce and interpret descriptive statements.[31] Thus, language and what it could do for us was held in an extremely positive light for Popper given its built-in or dispositional nature.[32] This was a source of optimism for Popper, both in the possibility for individual progress and flourishing, but also in civilizational and societal progress. We have inbuilt mechanisms that enable us to understand what others really mean and to use this in social problem solving to improve society. For Popper, such built-in dispositional knowledge is largely unconscious and often comes in the form of *expectations* and *attempted adaptations*. Despite Popper's emphasis upon linguistic communication, particularly the hypothetical-deductive problem solving of his theory of conjectures and refutations, he argued that 99 percent of the knowledge of all organisms, including humans, is inborn and incorporated in our biochemical constitution.[33] So while Popper only rarely used the term intuition or "intellectual intuition", he readily referred to "inborn" knowledge that unconsciously guides our actions such as our expectations.

Popper does not oppose World 2 constructivism, rather the emergence of mathematics in World 2 continues 'autonomously' in World 3 via our ability to criticise and discuss such initially extra-linguistic mathematical objects. Thus for Popper, not only is the general doctrine of intuition as an infallible source of knowledge regarded as erroneous, but our intuition of time, more especially, is just as subject to criticism and correction as well as by Brouwer's own admission, our intuition of space. Popper's main idea here was one which, he stated that he owed to Lakatos's philosophy of mathematics. For Lakatos, mathematics (and not only the natural sciences) grows through the criticism of guesses, and bold informal proofs. This for Popper, presupposes the linguistic formulation of these guesses and proofs, and thus their status in "the third world". For Popper:

Language, at first merely a means of communicating descriptions of prelinguistic objects, becomes thereby an *essential part* of the scientific enterprise, even in mathematics, which in its turn becomes part of the third world. And there are layers, or levels, in language (whether or not they are formalized in a hierarchy of metalanguages).[34]

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[31] Popper, *Unended Quest*, pp. 50-51.

[32] Popper stated that background knowledge "may even be built into the grammar of our language – and, therefore, like the air we breathe, constantly assumed or presupposed in our arguments, so that it is difficult for us to detect it and criticise it." See: Popper, *Knowledge and The Mind-Body Problem: In Defence of Interaction*, p. 135.

[33] See: Popper, "Towards and Evolutionary Theory of Knowledge", in *A World of Propensities*, p. 46.

[34] Ibid., p. 136.

It is for this reason that Popper regard as mistaken Brouwer's subjectivist epistemology and the philosophical justification of his intuitionist mathematics. He argued that there is a give-and-take between construction, criticism, 'intuition', and even tradition, which he fails to consider.[35] He was, however, prepared to admit that even in Brouwer's view of the status of language that he regarded erroneous, Brouwer was nonetheless partly right. That although the objectivity of all science, including mathematics, is inseparably linked with its criticizability, and therefore with its linguistic formulation, "Brouwer was right in reacting strongly against the thesis that mathematics is *nothing but* a formal language game or, in other words, that there are no such things as extra-linguistic mathematical objects". Brouwer insisted, mathematical talk is *about* these objects; and in this sense, mathematical language is secondary to these objects. But this for Popper did not mean that we could construct mathematics without language as "there can be no construction without constant critical control, and no criticism without putting our constructs into a linguistic form and treating them as objects of the third world."[36] While both Popper and Brouwer accepted intellectual intuition, they disagreed on the function, scope and characteristics of this kind of knowledge, particularly for the domain of mathematics.

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[35] Ibid., p. 137.
[36] Ibid., p. 137.

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